

Worksheet 4.11 Counting

Section 1 INTRODUCTION

When looking at situations involving counting it is often not practical to count things individually. Instead techniques have been developed to help us count efficiently and accurately. In particular this chapter looks at permutations and combinations.

Firstly however we must look at the Fundamental Principle of Counting (sometimes referred to as the multiplication rule) which states:

If there are m ways of doing one thing and n ways of doing another, then there are $m \times n$ ways of doing the first thing followed by the second.

This rule can best be understood by looking at an example.

Example 1 : There are 3 t-shirts and 2 pairs of jeans in the cupboard. How many possible outfits are there?

For each t-shirt there are 2 possibilities of jeans. All together we have three lots of these two possibilities, or, $3 \times 2 = 6$.

Example 2 : An ice-cream shop offers 3 types of cones and 5 different flavours of ice-cream. How many possible ice-cream cone combinations are there?

For each of the 3 cones there are 5 possible toppings so altogether there are $3 \times 5 = 15$ possible ice-cream cone combinations.

Exercises:

1. The local pizzeria offers a choice of 2 pizzas - supreme or vegetarian, 3 sides - chips, salad or coleslaw, and 4 drinks - juice, coke, ginger beer or water. For dinner I decide to have 1 pizza, 1 side, and 1 drink. How many possible meals do I have to choose from?
2. How many different car number plates can be made if each is to display 3 letters followed by 3 numbers?
3. Your friend wants to perform a magic trick and asks you to draw 2 cards from a standard deck of 52. The first card you draw must be placed face down and the second placed face up on the table. How many ways are there of drawing the 2 cards?

Section 2 COUNTING TECHNIQUES

For the many circumstances where we need to count the number of outcomes there are two different counting situations - permutations and combinations. A permutation is an arrangement where the order of selection matters. A combination is an arrangement where the order of selection doesn't matter.

Example 1 : The number of ways of arranging 5 people in a line. There are 5 choices for the first spot, 4 choices for the second and so on, so we have $5 \times 4 \times 3 \times 2 \times 1 = 5!$. Therefore the number of ways of arranging n things into n places is $n!$.

Example 2 : To arrange 3 people out of 5 in a line there are $5 \times 4 \times 3 = \frac{5!}{2!}$ ways. These are all permutations, the number of ways of choosing k things out of n where the order matters is

$$= \frac{n!}{(n-k)!} = {}^n P_k.$$

How about choosing 3 out of 5 people to put on a committee? This time, it doesn't matter whether you are chosen first, second or third, you are still on the committee. There are $5 \times 4 \times 3$ ways of choosing 3 people in order. However since order doesn't matter we have over counted, and therefore we need to divide by the number of ways of arranging the three people that are chosen, $3!$. So the number of ways of choosing this committee is the number of ways of choosing with order, divided by $3 \times 2 \times 1$.

$$\frac{5 \times 4 \times 3}{3 \times 2 \times 1} = \frac{5!}{2!3!}.$$

This is a combination, the number of ways of choosing k things from n where order does **NOT** matter

$$= \frac{n!}{k!(n-k)!} = {}^n C_k.$$

Definition 1 : A permutation is the number of ways of choosing k things out of a possible n , where the order that they are chosen matters and is notated ${}^n P_k$.

$${}^n P_k = n(n-1)(n-2)\dots(n-k+1) = \frac{n!}{(n-k)!}.$$

Definition 2 : A combination is the number of ways of choosing k things out of a possible n , where the order that they are chosen does not matter and is notated ${}^n C_k$.

$${}^n C_k = \frac{n(n-1)(n-2)\dots(n-k+1)}{k(k-1)\dots 1} = \frac{n!}{(n-k)!k!}.$$

These definitions are best illustrated by examples which we'll cover below and in the next 2 sections.

Example 3 :

- (a) A teacher wants to randomly choose 5 people from the class of 30 to help out at the open day BBQ. In how many ways can this be done?
- (b) A teacher wants to award prizes for 1st, 2nd, 3rd, 4th and 5th in the class of 30. In how many ways can the prizes be awarded (assume no two students tie)?

Answers:

- (a) In the first case, the 5 people are chosen, and it doesn't matter whether a person is chosen first, second or fifth, they all receive the same extra work, and so there are

$${}^{30}C_5 = \frac{30 \times 29 \times 28 \times 27 \times 26}{5 \times 4 \times 3 \times 2 \times 1} = 142,506$$

ways to choose the students with those conditions.

- (b) Here the order that the student are chosen does matter, and so there are

$${}^{30}P_5 = 30 \times 29 \times 28 \times 27 \times 26 = 17,100,720$$

ways to choose the students with those conditions. This is a significantly more as expected.

Notice the short way of calculating ${}^{30}P_5$ and ${}^{30}C_5$ applies to all such calculations. To find ${}^{30}P_5$ you can just multiply 5 consecutive descending numbers together, starting at 30. Similarly calculating ${}^{30}C_5$ can be done by multiplying 5 consecutive descending numbers together, starting at 30, and then dividing by the product of 1 through 5.

Exercises:

1. Calculate ${}^{10}P_4$ and ${}^{10}C_4$.
2. Calculate 7P_5 and 7C_2 .
3. In an 18 team league, how many ways can the 8 teams for the finals be decided? In how many ways can the first 4 positions be decided?
4. A child wants to draw a picture using only three different colours from a set containing twelve different colour pencils. In how many ways can the colours be chosen?
5. In the front of a building there are three doors each to be painted a different colour from twelve different available colours. How many colour arrangements for the doors are there?

Section 3 PERMUTATIONS

Example 1 :

- (a) How many four digit numbers can be formed using only the digits 1, 2, 3, 4, 5, 6?
- (b) How many four digit numbers from (a) have no repeated digits?
- (c) How many four digit numbers from (b) are greater than 5000?

Answers:

- (a) There are 6 possible digits for each of the four places in the number, so there are $6 \times 6 \times 6 \times 6 = 6^4 = 1296$ of these numbers.
- (b) There are 6 digits for the first place, and then only 5 digits for the second and so on. So there are $6 \times 5 \times 4 \times 3 = 360$ such numbers.
- (c) There are 2 choices for the first digit (a 5 or a 6), then 5 choices, 4 choices and 3 choices respectively for the remaining digits. So there are $2 \times 5 \times 4 \times 3 = 120$ such numbers.

Example 2 : Three adults and five children are seated randomly in a row.

- (a) In how many ways can this be done?
- (b) In how many ways can this be done if the three adults are seated together?
- (c) In how many ways can this be done if the three adults are seated together and the five children are also seated together.

Answers:

- (a) There are $8!$ ways of arranging 8 people in a row.
- (b) There are $3!$ ways of arranging the adults. We now need to arrange 6 objects (1 group of adults and 5 individual children) in a row. Therefore the answer is $3! \times 6!$ ways.
- (c) There are $3!$ ways of arranging the adults, $5!$ ways of arranging the children, and $2!$ ways of arranging the 2 groups. So the answer is $3! \times 5! \times 2!$.

Permutations with Repeated Objects

Example 3 : How many arrangements of the letters of the word IRRIGATION are there?

Answer: There are 10 letters. If the letters were all different there would be $10!$ arrangements. However there are three I's and two R's and so we need to divide by $3! \times 2!$. Therefore the answer is $\frac{10!}{3!2!}$.

Example 4 : In how many ways can we rearrange the letters in "MATHS IS FUN"

- (a) with no restrictions?
- (b) if the first and last letter must be vowels?

Answers:

- (a) There are 10 letters to rearrange. Two of them are S's and so that we do not over count we need to divide by the number of ways of arranging the S', so there are $\frac{10!}{2!}$ ways to arrange these letters.
- (b) In the second instance, we first select the vowels and there are 3×2 ways of selecting the first and last as vowels. Then we have 8 letters left with 2 S's repeating so there are $\frac{8!}{2!}$ ways to do this. Hence there are $3 \times 2 \times \frac{8!}{2!}$ ways of arranging these letters so that the first and last are vowels.

Exercises:

1. How many 4 digit numbers can be formed from the digits 1,2,3,4 if
 - (a) repetitions are allowed?

- (b) repetitions are not allowed?
2. How many 3 digit arrangements can be formed from the digits 0,1,2,3,4,5,6,7 and 8?
 3. Seven people are to occupy consecutive seats in a theatre. In how many ways can this be done if
 - (a) there are no restrictions?
 - (b) two people A and B sit at opposite ends of the row of seven seats?
 - (c) two people A and B sit together?
 - (d) two people A and B do not sit together?
 4. How many arrangements of the letters in the following words are possible?
 - (a) SYDNEY (b) GEOMETRY (c) EXCELLENCE
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Section 4 COMBINATIONS

Example 1 : A student must select 6 subjects. In how many ways can they do that if there are 13 subjects and 1 is compulsory?

Since one subject is compulsory the student must select 5 subjects from 12, there are

$${}^{12}C_5 = \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1} = 792,$$

ways to do this.

Example 2 : In a lottery you select 6 numbers out of 40, how many ways are there to do this?

You are selecting 6 things from 40 with order not mattering, thus there are

$${}^{40}C_6 = \frac{40 \times 39 \times 38 \times 37 \times 36}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 3,838,380 \text{ways.}$$

Example 3 : You want to choose a committee of 5 people from 7 men and 8 women.

- (a) How many ways can this be done?
- (b) How many ways can this be done if you want a majority of women on the committee?

In (a) it doesn't matter if the 5 people are men or women. So we are choosing 5 people from 15 and there are ${}^{15}C_5$ ways of doing this.

In (b) we need either 3 women (and 2 men) or 4 women (and 1 man) or 5 women. So there are ${}^8C_3 \times {}^7C_2 + {}^8C_4 \times {}^7C_1 + {}^8C_5$ ways of doing this.

Exercises:

1. In how many ways can you choose 2 chocolates from a bag containing 6 different chocolates?
2. Twelve dots are spaced equally around a circle. How many different triangles can be formed by joining dots?
3. Tickets are numbered from 1 to 25. 6 tickets are chosen. In how many ways can this be done if the selection contains
 - (a) all odd numbers?
 - (b) 3 odd numbers and 3 even numbers?
 - (c) the numbers 1 and 2?

Exercises for Worksheet 4.11

- Decide whether or not order of selection is important and then calculate the following.
 - How many different sets of three colours can be selected from the colours red, orange, yellow, green, blue, and violet?
 - In how many ways can a team of five basketball players be selected from 8 girls?
 - A race has 8 runners. In how many ways can the first three places be decided?
 - A secretary has nine letters and only five stamps. How many ways can he select the letters for posting?
- How many different possible “full house” (one pair, one three of a kind) hands are there in 5 card poker?
- We choose 12 cards from the usual deck of 52 playing cards.
 - How many different ways can this be done?
 - How many ways can it be done if they must all come from the same suit?
 - How many ways can it be done if we need exactly 3 kings and 3 queens?
 - How many ways can it be done if all cards must have different face values?
- There are 10 boys and 11 girls at a school.
 - How many different ways can the boys each choose a girl to take to the formal?
 - One of the girls doesn't want to go to the formal, how many ways are there to make this choice now?
- A town of 30 people is to choose a committee of 3 to represent them, how many different ways can this be done? How many ways can it be done if one person is to be the chairperson, one the treasurer and one the secretary?
- In a certain electorate there are 6 candidates: labor, liberal, greens, and three independents. Their names are to be placed in random order on the ballot paper. In how many ways can this be done if
 - the labor candidate comes first?
 - the liberal candidate comes first?
 - the three independent candidates are together?
- From a class of 30 students, five students are to be selected to complete a survey. In how many ways can the choice be made?

8. A 4 digit password is to be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. How many different passwords are possible
- (a) if the digits are repeated?
 - (b) if there are no repeated digits?
9. How many different arrangements of the word ELLIPSE are possible if
- (a) there are no restrictions?
 - (b) the arrangement starts with S?
 - (c) both L's are together?
 - (d) the letters are in alphabetical order?
10. Seven people sit in a circle. How many ways can this be done if
- (a) there are no restrictions?
 - (b) two people A and B sit together?
 - (c) three people A, B and C sit together?
11. A committee of 5 is to be chosen from 4 men and 6 women. In how many ways can this be done if
- (a) there are no restrictions?
 - (b) the committee consists of women only?
 - (c) there is at least one man?
 - (d) there is a majority of women?
12. 5 cards are to be chosen from a standard 52-card deck. In how many ways can this be done if
- (a) all of the cards are clubs?
 - (b) all of the cards are of the same suit?
 - (c) there are three clubs and two spades?
 - (d) there are three of one suit and two of another?
13. A bag contains 5 red, 6 blue and 4 yellow marbles. Three are drawn out at random. In how many ways can they be drawn so that
- (a) are all blue?
 - (b) are all the same colour?
 - (c) are all different colours?

14. In order to be photographed 10 people stand in two rows of 5, one in front of the other. In how many ways can this be done if
- (a) there are no restrictions?
 - (b) A and B are to be in the front row?
 - (c) A and B are in different rows?
15. In how many ways can you choose 10 people out of 30 to sit on a bench in order, if two particular people must be selected and seated together?
16. You are to pick three singers for the Superbowl, one to sing before the game, one to sing at the end of the game, and one to sing at halftime. If there are 12 applicants, how many different possibilities are there if each singer can only sing once?

Answers for Worksheet 4.11

Section 1

1. 24

2. $26^3 \times 10^3$

3. 2652

Section 2

1. 5040, 210

3. 43758, 73440

5. ${}^{12}P_3$

2. 2520, 21

4. ${}^{12}C_3$

Section 3

1. (a) 256

(b) 24

2. 504

3. (a) 5040

(b) 240

(c) 1440

(d) 3600

4. (a) 360

(b) 20160

(c) 37800

Section 4

1. 6C_2

3. (a) ${}^{13}C_6$

(b) ${}^{13}C_3 \times {}^{12}C_3$

2. ${}^{12}C_3$

(c) ${}^{23}C_4$

Exercises 4.11

1. (a) Not important, 6C_3 (c) Important, 8P_3
 (b) Not important, 8C_5 (d) Important (if stamps are distinct), 9P_5
2. ${}^{13}C_1 \times {}^4C_3 \times {}^{12}C_1 \times {}^4C_2$
3. (a) ${}^{52}C_{12}$ (b) ${}^4C_1 \times {}^{13}C_{12}$ (c) ${}^4C_3 \times {}^4C_3 \times {}^{44}C_6$ (d) ${}^{13}C_{12} \times 4^{12}$
4. (a) ${}^{11}P_{10}$ (b) $10!$
5. ${}^{30}C_3, {}^{30}P_3$
6. (a) $5!$ (b) $5!$ (c) $3! \times 4!$
7. ${}^{30}C_5$
8. (a) 10^4 (b) ${}^{10}P_4$
9. (a) $\frac{7!}{2!2!}$ (b) $\frac{6!}{2!2!}$ (c) $\frac{6!}{2!}$ (d) $2! \times 2!$
10. (a) $\frac{7!}{7}$ (b) $2! \times \frac{6!}{6}$ (c) $3! \times \frac{5!}{5}$
11. (a) ${}^{10}C_5$ (b) 6C_5 (c) 246 (d) 186
12. (a) ${}^{13}C_5$ (c) ${}^{13}C_3 \times {}^{13}C_2$
 (b) ${}^4C_1 \times {}^{13}C_5$ (d) ${}^4C_1 \times {}^{13}C_3 \times {}^3C_1 \times {}^{13}C_2$
13. (a) 6C_3 (b) ${}^5C_3 + {}^6C_3 + {}^4C_3$ (c) $5 \times 6 \times 4$
14. (a) $10!$ (b) ${}^5P_2 \times 8!$ (c) $2 \times 5 \times 5 \times 8!$
15. $2! \times {}^{28}C_8 \times 9!$
16. ${}^{12}C_3 \times 3!$